

# Quasiparticle Spectrum Near the Gap Node Directions in the Mixed State of D-wave Superconductors

A. S. Mel'nikov

*Institute for Physics of Microstructures, Russian Academy of Sciences  
603600, Nizhny Novgorod, GSP-105, Russia*

(February 7, 2008)

## Abstract

We report on a calculation of the quantized energy spectrum and quasiparticle eigenfunctions for low lying excitations in the mixed state of clean d-wave superconductors. Our study is based on an approximate analytical solution of the Bogolubov-de Gennes equations for both rectangular and triangular flux lattices with one of the primitive translations chosen parallel to the gap node direction. For excitations with momenta close to a certain gap node we have obtained a set of eigenfunctions which appear to be extended along the chosen gap node direction and localized along the perpendicular one on a scale determined by the intervortex distance. The periodic superfluid velocity field induces a band structure in the spectrum, which depends essentially on the vortex lattice geometry.

PACS numbers: 74.80.Dm, 74.20.De, 74.60.Ge

Recently a great deal of attention has been devoted to the nature of the quasiparticle states in isolated vortices and vortex lattices in superconductors with anisotropic pairing. This problem is of considerable importance since low-energy quasiparticle excitations influence strongly on various static and dynamic properties of the mixed state at low temperatures. These investigations were stimulated by a large number of experiments which provide good evidence for the d-wave symmetry of the order parameter in high-temperature superconductors (see [1] and references therein). For conventional s-wave superconductors the low-lying quasiparticle states are bound to the vortex cores as it was first predicted by Caroli, de Gennes, and Matricon [2]. For an isolated vortex line the eigenvalues for these localized states may be written as follows:  $E_\mu \sim \mu\Delta/(k_F\xi)$ , where  $\Delta$  is the gap value far from the vortex axis,  $\xi$  is the coherence length at  $T = 0$ ,  $k_F$  is the Fermi momentum, and the angular momentum quantum number  $\mu$  is half an odd integer.

In contrast to conventional superconductors, the density of states (DOS) at low energies in d-wave systems is dominated by contributions which come from the regions far from the vortex cores and depends essentially on the intervortex distance  $R_v$ . This unusual fact is a direct consequence of the vanishing pair potential in directions of the gap nodes. As a result, one obtains the specific magnetic field dependence of DOS at the Fermi level (see Refs. [3–5]), which may be experimentally identified, for instance, in specific heat [6,7] or scanning tunneling microscope (STM) [8] measurements. It should be also emphasized that there is an important difference between the quantization of the energy spectrum in the mixed state of s- and d-wave superconductors: for s-wave systems the energy quantization and corresponding localized states exist even in a single isolated vortex line (and they are weekly influenced by the presence of neibouring vortices at least for  $R_v \gg \xi$ ) while for the d-wave case the low lying energy spectrum may be quantized only due to the finite intervortex distance (see Refs. [9–13]). The latter conclusion is proved by the numerical solution of the Bogolubov-de Gennes (BdG) equations [14] which shows that there are no truly localized states for a single isolated vortex in a pure d-wave superconductor.

The most simple theoretical description of the excitation spectrum is based on the semiclassical approach which takes account of the Doppler shift of the quasiparticle energy by the local superfluid velocity  $\mathbf{V}_s$ :

$$\varepsilon(\mathbf{k}, \mathbf{r}) = \pm \sqrt{\hbar^2 V_F^2 (k - k_F)^2 + \Delta^2(\mathbf{k})} + \hbar \mathbf{k} \mathbf{V}_s(\mathbf{r}) \quad (1)$$

where  $V_F$  is the Fermi velocity. Hereafter we assume the Fermi surface to be two-dimensional, which is appropriate to high- $T_c$  superconductors, and take the gap function in the form  $\Delta_d = \Delta_0 k_x k_y / k_F^2$  (the  $x$  axis is taken along the [110] crystal direction and thus makes an angle  $\pi/4$  with the  $a$  axis of the  $CuO_2$  planes). We restrict ourselves to the study of a vortex lattice under a magnetic field applied parallel to the  $c$  axis and, as a consequence, the vector  $\mathbf{V}_s$  is a periodic function of  $x, y$  with the periodicity of the vortex lattice. One can separate two length scales for quasiparticle wavefunctions: an atomic length scale  $k_F^{-1}$  and a characteristic wavelength of a slowly varying envelope  $l$ . The semiclassical procedure is in general correct when the order parameter and superfluid velocity are modulated on a scale  $\Lambda \gg l$ . For s-wave superconductors we have  $l \sim \xi$  (the minimum spatial extension of wave packets made with excitations (1)) and, as a result, the semiclassical approach fails only for the vortex core region. For d-wave systems the  $l$  value appears to be angular dependent. In

in the homogeneous case the spectrum for the low-lying excitations which are close to one of the gap nodes (which corresponds, e.g., to the point  $\mathbf{k} = (k_F, 0)$ ) has the form:

$$\varepsilon(\mathbf{k}) \simeq \pm \Delta_0 \sqrt{\xi^2(k - k_f)^2 + k_y^2/k_F^2}, \quad (2)$$

where  $\xi = \hbar V_F / \Delta_0$ . The typical momenta in the  $x$  direction are  $q_x \sim \varepsilon / (\Delta_0 \xi)$  and the corresponding wavelength is of the order  $l_x \sim \Delta_0 \xi / \varepsilon$ . Comparing the  $l_x$  value with the intervortex distance  $R_v$  (the characteristic length of the superfluid velocity variation), one can expect that the semiclassical approach based on the expression (1) fails for low energies  $\varepsilon \lesssim \Delta_0 \xi / R_v$ . In this case we may assume that the effective potential  $\hbar \mathbf{k} \mathbf{V}_s(\mathbf{r})$  in Eq. (1) should be averaged over the distances  $\sim l_x$  in the  $x$  direction (the validity of these qualitative arguments will be proved below). To analyse the problem beyond the semiclassical approach one must use the more powerful methods based on either the BdG equations or Green's-function techniques (which are equivalent for clean superconductors). The quasiclassical limit ( $k_F \xi \gg 1$ ) of these theories is known to be represented by the Andreev's and Eilenberger equations, respectively. Within these models one should solve the one-dimensional quantum mechanical problem for the particle motion along the quasiclassical trajectory which is characterized by the impact parameter  $b = \mu / k_F$  and the angle  $\theta$  in the  $x - y$  plane [3,9–12,15,16]. Using the Bohr-Sommerfeld quantization rule for the angular momentum  $\mu(\theta, \varepsilon)$  one can in principle determine the true quantum levels (see Refs. [9–12]). For the d-wave case the main difficulty is connected with the description of quasiclassical trajectories which make an angle  $\theta < \xi / R_v$  with the gap node directions. For this angular domain the extension of the wave function exceeds the  $R_v$  value and quasiparticle states are sensitive to the superfluid velocity fields of all vortices even if the impact parameter is less than  $R_v$ . The correct description of these trajectories with small  $\theta$  values is of considerable importance since it is this angular interval which determines the true quantum levels according to the Bohr-Sommerfeld quantization rule.

It is the purpose of this Letter to report on a calculation of the quantized energy spectrum and quasiparticle eigenfunctions for various vortex lattice structures, based on an approximate analytical solution of the BdG equations for low lying excitations with  $\varepsilon \lesssim \Delta_0 \xi / R_v$ . Our approach provides a possibility to take proper account of the influence of the superfluid velocity fields of all vortices in the vortex lattice on the quasiparticle motion along the trajectories almost parallel to gap node directions. We consider the case of pure d-wave superconductivity and neglect the effects connected with the possible formation of states of mixed symmetry (with coexisting s- and d-wave or  $d_{x^2-y^2}$  and  $d_{xy}$  order parameter components). The BdG equations can be written as:

$$\hat{h}_0 u(\mathbf{r}) + \int \Delta(\mathbf{r}, \mathbf{r}') v(\mathbf{r}') d\mathbf{r}' = \varepsilon u(\mathbf{r}) \quad (3)$$

$$- \hat{h}_0^* v(\mathbf{r}) + \int \Delta^*(\mathbf{r}, \mathbf{r}') u(\mathbf{r}') d\mathbf{r}' = \varepsilon v(\mathbf{r}) \quad (4)$$

where  $u, v$  are the particlelike and holelike parts of the quasiparticle wave function. The one-particle Hamiltonian  $\hat{h}_0$  in the most simple isotropic case takes the form

$$\hat{h}_0 = -\frac{\hbar^2}{2m} \left( \nabla + i \frac{\pi}{\phi_0} \mathbf{A} \right)^2 - E_F,$$

where  $\phi_0$  is the flux quantum and  $E_F$  is the Fermi energy. The system (3,4) has been previously solved numerically for specific lattice models [17,18] and in the continuum limit [14,19]. To obtain the analytical solution we follow the well known procedure (see Refs. [13,20]) and simplify the nonlocal off-diagonal terms using the condition  $k_F\xi \gg 1$  (or, equivalently,  $\Delta_0 \ll E_F$ ). In this case one can search for the solution in the form  $u = U \exp(i\mathbf{k}_F \mathbf{r})$ ,  $v = V \exp(i\mathbf{k}_F \mathbf{r})$ , i.e. divide out the fast oscillations on a scale  $k_F^{-1}$ . Then we rewrite  $\Delta(\mathbf{r}, \mathbf{r}')$  in terms of the center of mass  $\mathbf{R} = (\mathbf{r} + \mathbf{r}')/2$  and relative coordinates  $\rho = \mathbf{r} - \mathbf{r}'$  and introduce the gap function as a Fourier transform with respect to  $\rho$ :  $\Delta(\mathbf{k}, \mathbf{R}) = \Delta_d(\mathbf{k})\Psi(\mathbf{R})$ . The function  $\Psi(\mathbf{R}) = f \exp(i\chi)$  is the d-wave order parameter used in Ginzburg-Landau theory. Let us take the two-term expansion for the gap operator:

$$\int \Delta(\mathbf{r}, \mathbf{r}') v(\mathbf{r}') d\mathbf{r}' \simeq e^{i\mathbf{k}_F \mathbf{r}} \left( \Psi \Delta_d(\mathbf{k}_F) V - \frac{i}{2} \frac{\partial \Delta_d}{\partial \mathbf{k}} \Big|_{\mathbf{k}_F} \{\nabla, \Psi\} V \right), \quad (5)$$

where we use the notation  $\{\hat{A}, \hat{B}\}$  for the anti-commutator of two operators  $\hat{A}$  and  $\hat{B}$ . In order to obtain the Andreev's equations we should keep only the first term in this expression. Obviously such an approximation is not correct and the second term can not be omitted when  $\mathbf{k}_F$  is close to the gap nodes and the first term vanishes. Taking, e.g.,  $\mathbf{k}_{F1} = (k_F, 0)$  and introducing a new two-component wave function  $\hat{g} = (U \exp(-i\chi), V)$  (to eliminate the order parameter phase in  $\Psi$ ) one obtains the equations linearized in gradient terms:

$$\begin{aligned} & -\xi \hat{\sigma}_z \left( i \frac{\partial}{\partial x} + \frac{\pi A_x}{\phi_0} \right) \hat{g} - \frac{\hat{\sigma}_x}{2k_F} \left\{ i \frac{\partial}{\partial y} + \frac{\pi A_y}{\phi_0}, f \right\} \hat{g} \\ & + \frac{m\xi}{\hbar} V_{sx} (1 + \hat{\sigma}_z) \hat{g} + \frac{mf}{\hbar k_F} V_{sy} \hat{\sigma}_x \hat{g} = \frac{\varepsilon}{\Delta_0} \hat{g}, \end{aligned} \quad (6)$$

where  $\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z$  are the Pauli matrices. We follow here the treatment in Ref. [13] and neglect the curvature of the Fermi surface. For an isotropic Fermi surface such an approximation is valid only for  $\varepsilon \ll \Delta_0/(k_F\xi)$  (see Refs. [21,22]). However, as mentioned in [13,22], the range of validity of Eq. (6) may be even larger if the Fermi surface is somewhat flattened at the nodes. At intermediate magnetic fields  $H_{c1} \ll H \ll H_{c2}$  (when the intervortex distance is much smaller than the London penetration depth) we can assume  $\mathbf{H} = -H \mathbf{z}_0$  to be homogeneous and take the gauge  $\mathbf{A} = H y \mathbf{x}_0$ . Here  $\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0$  are the unit vectors of the coordinate system. For a vortex lattice with primitive translations  $\mathbf{a}_1, \mathbf{a}_2$  the superfluid velocity may be written in the form:

$$\mathbf{V}_s = \frac{i\pi\hbar H}{m\phi_0} \sum_{\mathbf{b} \neq 0} \frac{[\mathbf{b}, \mathbf{z}_0]}{|\mathbf{b}|^2} e^{i\mathbf{b}\mathbf{r}}, \quad (7)$$

where  $\mathbf{b} = n\mathbf{b}_1 + m\mathbf{b}_2$  and  $\mathbf{b}_1 = 2\pi H[\mathbf{a}_2, \mathbf{z}_0]/\phi_0$ ,  $\mathbf{b}_2 = 2\pi H[\mathbf{z}_0, \mathbf{a}_1]/\phi_0$  are the primitive translations in the reciprocal lattice,  $n$  and  $m$  run over all possible integers. As we see below the solution of Eq. (6) depends strongly on the flux lattice structure and its orientation relative to the crystal axes. Contrary to the conventional isotropic superconductors (where a hexagonal flux lattice appears to be energetically favourable), for d-wave compounds previous theoretical works predicted a rich phase diagram, containing triangular,

centered rectangular and square lattices with various orientations relative to the ionic lattice, as a function of magnetic field and temperature [23–28]. In principle one should treat the problem self-consistently, i.e. calculate at first the energy spectra for various vortex configurations and then find the lattice structure corresponding to the free energy minimum.

In this paper we do not solve this self-consistent problem and restrict ourselves to the study of energy spectra for several particular lattice structures. Let us choose one of the primitive translations (e.g.  $\mathbf{a}_1$ ) to be parallel to the gap node direction  $\mathbf{x}_0$  and consider two types of lattices: (I)  $\mathbf{a}_1 = a\mathbf{x}_0$ ,  $\mathbf{a}_2 = \sigma a\mathbf{y}_0$ ,  $H\sigma a^2 = \phi_0$  (rectangular lattice); (II)  $\mathbf{a}_1 = a\mathbf{x}_0$ ,  $\mathbf{a}_2 = a(\mathbf{x}_0/2 - \sigma\mathbf{y}_0)$ ,  $H\sigma a^2 = \phi_0$  (vortices in the unit cell form a shape of an isosceles triangle with the base along the  $x$  axis). Note that the centered rectangular lattices of the type II (with the gradually changing parameter  $\sigma$  as a function of  $H$  and  $T$  values) were found to be energetically favourable for a certain region of the  $H - T$  phase diagram within a generalized London model taking account of nonlocal and nonlinear corrections to the free energy [23,24]. We also include in our consideration the square lattice tilted by  $\pi/4$  from the  $a$  axis (type I,  $\sigma = 1$ ) which is most stable at least for rather high fields and temperatures not very close to  $T_c$  according to Refs. [25–28]. Such a lattice structure (though elongated in the  $a$  direction) is close to the one observed experimentally in twinned YBaCuO monocrystals by small angle neutron scattering [29] and scanning tunneling microscopy [8].

If we search for the solution of Eq. (6) as a linear combination of harmonics  $e^{iq_x x} \hat{G}(q_x, y)$ , the periodic functions  $f$  and  $\mathbf{V}_s$  will be responsible for the interaction of harmonics with  $q_x$  and  $q_x + nb_{1x}$ . As we see below, for  $\varepsilon < 0.5\pi\Delta_0\xi/a$  the wavefunctions  $\hat{G}(q_x, y)$  and  $\hat{G}(q_x + nb_{1x}, y)$  do not overlap in the  $y$  direction and, consequently, their interaction is negligible. In this case the  $q_x$  momentum component is a good quantum number and one can replace the exact periodic potential (7) in Eq. (6) by the effective potential averaged in the  $x$  direction. The equation for  $\hat{G}(q_x, y)$  reads:

$$\xi\hat{\sigma}_z \left( q_x - \frac{\pi H y}{\phi_0} \right) \hat{G} - \frac{i}{k_F} \hat{\sigma}_x \frac{\partial \hat{G}}{\partial y} + \frac{\pi\xi}{2} \sqrt{\frac{\sigma H}{\phi_0}} \Phi \left( y\sqrt{\frac{H}{\sigma\phi_0}} \right) (1 + \hat{\sigma}_z) \hat{G} = \frac{\varepsilon}{\Delta_0} \hat{G}, \quad (8)$$

where  $\Phi(z) = 2z - (2m + 1)$  for  $m < z < m + 1$ ,  $m \in \mathbb{Z}$ . We omitted here the small corrections of the order  $\xi/a$ , which are connected with the suppression of the order parameter  $f$  in vortex cores. It is convenient to introduce  $\hat{F} = (\hat{\sigma}_x + \hat{\sigma}_z)\hat{G}/2$  and dimensionless variables  $y\sqrt{H/(\sigma\phi_0)} = z$ ,  $q_x = \pi Q\sqrt{H\sigma/\phi_0}$ ,  $\varepsilon = E\Delta_0\pi\xi\sqrt{H\sigma/\phi_0}$ . At the  $m$ -th interval ( $m < z < m + 1$ ) one obtains:

$$-i\lambda\hat{\sigma}_z \frac{\partial \hat{F}}{\partial z} + \left( \frac{1}{2}\Phi(z) - E \right) \hat{F} + q_m \hat{\sigma}_x \hat{F} = 0, \quad (9)$$

where  $q_m = Q - m - 1/2$  and  $\lambda = (\pi\sigma k_F \xi)^{-1}$  is a dimensionless wavelength. The procedure analogous to the one used above for the gap node at the point  $\mathbf{k}_{F1} = (k_F, 0)$  can be carried out for the perpendicular gap node direction  $\mathbf{k}_{F2} = (0, k_F)$ . Taking the gauge  $\mathbf{A} = -Hxy\mathbf{y}_0$  and introducing  $\hat{F} = (\hat{\sigma}_x + \hat{\sigma}_z)\hat{G}^*/2$ ,  $-x\sqrt{H\sigma/\phi_0} = z$ ,  $q_y = \pi Q\sqrt{H/(\sigma\phi_0)}$ ,  $\varepsilon = E\Delta_0\pi\xi\sqrt{H/(\sigma\phi_0)}$ ,  $\lambda = \sigma(\pi k_F \xi)^{-1}$  ( $-2x\sqrt{H\sigma/\phi_0} = z$ ,  $2q_y = \pi Q\sqrt{H/(\sigma\phi_0)}$ ,  $2\varepsilon = E\Delta_0\pi\xi\sqrt{H/(\sigma\phi_0)}$ ,  $\lambda = 4\sigma(\pi k_F \xi)^{-1}$ ) for the type I (II) lattices one obtains Eq. (9). The symmetry of BdG equations allows us to conclude that the solutions  $U_{3,4}, V_{3,4}, \varepsilon_{3,4}$  for two

other gap nodes at the points  $\mathbf{k}_{F3} = (-k_F, 0)$  and  $\mathbf{k}_{F4} = (0, -k_F)$  can be found by simple transformation:  $U_{3,4} = V_{1,2}^*$ ,  $V_{3,4} = -U_{1,2}^*$ ,  $\varepsilon_{3,4} = -\varepsilon_{1,2}$ .

It may be useful to note that inside the  $m$ -th interval the equations (9) are equivalent to the ones describing the interband tunneling [30] or the one-dimensional (1D) motion of a Dirac particle in a uniform electric field and can be solved exactly in terms of parabolic cylinder functions. To obtain the energy spectrum one should match these solutions at points  $z = m$ . This procedure may be essentially simplified using the usual 1D quasiclassical approach which is valid for  $\lambda \ll 1$ . Substituting  $\hat{F} \propto \exp(iS(z))$  in Eq. (9) we obtain the classically allowed (CA) regions (where the function  $S$  is real):  $|\Phi(z)/2 - E| > |q_m|$ . These regions shift in the  $z$  direction with a change in the  $Q$  value and the quasiparticle motion at the  $m$ -th interval is classically allowed only for  $|q_m| < 1/2 + |E|$ . From the latter condition one can see that for rather low energies ( $|E| < 1/2$ ) the CA regions corresponding to the values  $Q$  and  $Q + 2n$  do not overlap. In this case the interaction of harmonics with  $q_x$  and  $q_x + nb_{1x}$  will be exponentially small due to the exponential decay of the wavefunction in classically forbidden regions. Thus, our procedure based on the averaging of the superfluid velocity in the  $x$  direction is correct only for low energies  $|E| < 1/2$ , when the solutions are essentially localized in the  $y$  direction on a scale determined by the intervortex distance.

Let us now consider the domain  $|q_m| < 1$  and continue with the calculation of quasiclassical wavefunctions and energy levels corresponding to the quasiparticle motion at the  $m$ -th interval which contains two CA regions: (i)  $-1/2 < z - m - 1/2 < E - |q_m|$  (region A) and (ii)  $E + |q_m| < z - m - 1/2 < 1/2$  (region B). Note that the region A (B) exists if the turning point  $z_{1m} = m + 1/2 + E - |q_m|$  ( $z_{2m} = m + 1/2 + E + |q_m|$ ) belongs to the  $m$ -th interval. The wavefunctions and quantization rules for regions A and B can be written as follows:

$$\begin{aligned} \hat{F}_{A,B} &= \frac{C_{A,B}}{(t^2 - q_m^2)^{1/4}} \left( \frac{\exp(iS_{A,B})}{|t + \sqrt{t^2 - q_m^2}|^{1/2}} \left( t + \sqrt{t^2 - q_m^2} \right) + \right. \\ &\quad \left. + \frac{\exp(-iS_{A,B})}{|t - \sqrt{t^2 - q_m^2}|^{1/2}} \left( t - \sqrt{t^2 - q_m^2} \right) \right) \\ S_{A,B} &= \frac{1}{\lambda} \int_{\mp|q_m|}^t \sqrt{s^2 - q_m^2} ds \pm \frac{\pi}{4} \end{aligned} \quad (10)$$

$$\int_{|q_m|}^{1/2 \pm E} \sqrt{t^2 - q_m^2} dt = \pi \lambda (N_{A,B} + \gamma_{A,B}(q_m)) \quad (11)$$

where  $t = z - m - 1/2 - E$ ,  $N_{A,B}$  is an integer. The  $\gamma_{A,B}$  values are of the order unity and determined by the matching of the expression (10) with the exponentially decaying solutions in classically forbidden regions at the  $(m-1)$ -th and  $(m+1)$ -th intervals. This matching procedure results in the following boundary conditions for the wavefunction  $\hat{F} = (F_1, F_2)$ :

$$\left. \frac{F_2}{F_1} \right|_{z=m} = \exp(i\alpha_A); \quad \cos\alpha_A = \frac{E - 1/2}{Q - m + 1/2} \quad (12)$$

$$\left. \frac{F_2}{F_1} \right|_{z=m+1} = \exp(i\alpha_B); \quad \cos\alpha_B = \frac{E + 1/2}{Q - m - 3/2} \quad (13)$$

Evaluating the integral in the l.h.s. of Eq. (11) one obtains:

$$\left(\frac{1}{2} \pm E_{A,B}\right) \sqrt{\left(\frac{1}{2} \pm E_{A,B}\right)^2 - q_m^2} - q_m^2 \cosh^{-1} \left(\frac{\frac{1}{2} \pm E_{A,B}}{|q_m|}\right) = 2\pi\lambda(N_{A,B} + \gamma_{A,B}) \quad (14)$$

The important point is that a set of eigenvalues corresponding to a certain momentum  $Q$  coincides with that for the momentum  $Q + 1$ . Such a periodicity of the energy spectrum is a consequence of the periodicity of the potential  $\Phi(z)$  and can be proved exactly from Eq. (9). Thus, to analyse the spectrum one can consider only the  $Q$  values in the 1D Brillouin zone  $-1/2 < Q < 1/2$ . In Fig. 1 we display the energy branches (14) in the first Brillouin zone for the particular case  $\lambda = 0.01$ . Each energy branch  $E_A(m, N_A, Q)$  ( $E_B(m, N_B, Q)$ ) considered as a function of the momentum  $Q$  has a single minimum (maximum) at the point close to the Brillouin zone boundary  $Q = m + 1/2$ . Near the first Brillouin zone boundary  $|Q - 1/2| \ll 1/2 \pm E$  we obtain:

$$E_{A,B}|_{m=0} \simeq \mp \frac{1}{2} \pm \sqrt{2\pi\lambda(N_{A,B} + \gamma_{A,B})} \pm \frac{(Q - 1/2)^2}{2\sqrt{2\pi\lambda(N_{A,B} + \gamma_{A,B})}} \ln \frac{\sqrt{2\pi\lambda(N_{A,B} + \gamma_{A,B})}}{|Q - 1/2|} \quad (15)$$

Near the points of intersection of the energy branches (14) in the  $E - Q$  plane one should take account of the splitting of energy levels resulting from the quasiparticle tunneling through classically forbidden regions. As a result we obtain a set of narrow bands separated by energy gaps. The effect of tunneling (and, therefore, energy splitting) is exponentially small if the characteristic length of the wavefunction decay in a classically forbidden region is much smaller than the dimension of this region. In the opposite case the effect of tunneling is essential. In particular, we can not neglect the tunneling between the regions A and B for  $Q$  close to Brillouin zone boundaries, when the distance between the turning points  $z_{1m}$  and  $z_{2m}$  becomes comparable to the characteristic decay length  $\sqrt{\lambda}$  in the region  $z_{1m} < z < z_{2m}$ . To consider this limit let us take the exact solution at the  $m$ -th interval:

$$\hat{F} = \begin{pmatrix} AD_{-i\mu-1}(\tau) + BD_{-i\mu-1}(-\tau) \\ \text{sign}(Q - m - 1/2) \sqrt{i/\mu} (-AD_{-i\mu}(\tau) + BD_{-i\mu}(-\tau)) \end{pmatrix} \quad (16)$$

Here  $\tau = \frac{2}{i\lambda}(z - m - 1/2 - E)$ ,  $\mu = q_m^2/(2\lambda)$ ,  $D_{-i\mu-1}(\tau)$  and  $D_{-i\mu}(\tau)$  are the parabolic cylinder functions [31]. Using Eqs. (12,13,16) one obtains the system of two equations for  $A$  and  $B$ . The solvability condition for this system results in the quantization rule. Far from the turning points the parabolic cylinder functions can be replaced by the corresponding asymptotic expressions. Thus, if the turning points  $z_{1m}$  and  $z_{2m}$  are not too close to the  $m$ -interval boundaries ( $z_{1m} - m \gg |q_m|$ ,  $m + 1 - z_{2m} \gg |q_m|$ ) and the distance  $z_{2m} - z_{1m}$  is rather small ( $|q_m| \ll \sqrt{\lambda}$ ) then the quantization rule takes the following form:

$$E \simeq E_{0N} + \sqrt{\pi\lambda} q_m (-1)^N \sin \left( \frac{\pi}{4} + \frac{1}{4\lambda} + \frac{E_{0N}^2}{4\lambda} - \frac{q_m^2}{2\lambda} \ln \frac{(1/4 - E_{0N}^2)}{\lambda} - \frac{\alpha_A + \alpha_B}{2} \right) \quad (17)$$

$$E_{0N} = \pi N \lambda + \lambda(\alpha_A - \alpha_B)/2$$

where  $N \in Z$ . The spectrum appears to be equidistant only at the Brillouin zone boundaries ( $Q = \pm 1/2$ ) and the distance between the energy levels depends strongly on the vortex lattice geometry.

Using the solution of Eq. (9) given above we obtain four different sets of eigenfunctions and eigenvalues associated with four gap nodes. The range of validity of our approach is restricted by the conditions  $\varepsilon < 0.5\Delta_0\pi\xi\sqrt{H/\phi_0}\min[\sigma^{-1/2}, \sigma^{1/2}]$  and  $\varepsilon < 0.5\Delta_0\pi\xi\sqrt{H/\phi_0}\min[\sigma^{-1/2}/2, \sigma^{1/2}]$  for type I and II lattices, respectively. In the opposite limit the wavefunctions are not localized in the  $z$  direction and our approximate solution based on the averaging of the superfluid velocity in the gap node direction is no more valid. Thus, for quasiparticle states associated with each gap node  $\mathbf{k}_{\mathbf{F}\alpha}$  there exist two regimes with the crossover parameter  $\varepsilon R_{v\alpha}/(\Delta_0\xi)$  (where  $R_{v\alpha}$  is the intervortex distance in the gap node direction,  $\alpha = x, y$ ): (i) for the low energy regime the spectrum has a band structure and wavefunctions are extended in the gap node direction and localized in the perpendicular one; (ii) for the high energy regime the spectrum is continuous and eigenfunctions are extended in both directions. The characteristic energy  $\varepsilon$  of excitations coming into play at finite temperatures is of the order  $\sim T$  and, consequently, for thermodynamic and transport quantities one can expect different regimes with the crossover parameters  $TR_{vx}/(\Delta_0\xi)$ ,  $TR_{vy}/(\Delta_0\xi)$ . Taking the case  $R_{vx} \sim R_{vy} \sim \sqrt{\phi_0/H}$  we obtain the crossover parameter introduced previously in Refs. [9,11,21].

The quantization of the low energy spectrum should result in the oscillatory behavior of DOS as a function of energy with the characteristic energy scales of the order  $\delta\varepsilon_1 \sim \Delta_0\sqrt{\hbar\omega_c/(\sigma E_F)}$  and  $\delta\varepsilon_2 \sim \Delta_0\sqrt{\hbar\omega_c\sigma/E_F}$ , where  $\omega_c = eH/(mc)$  is the cyclotron frequency. Even if we neglect these oscillations and assume  $N$  to be a continuous variable (more detailed study is left as a future problem) the spatial and energy dependence of the local DOS in the low energy regime differs qualitatively from the one obtained from the semiclassical approach based on the expression (1). Taking account of contributions from four gap nodes we obtain:

$$N_I(\varepsilon, x, y) = \frac{N_F}{4}\sqrt{\frac{\pi H}{2H_{c2}}}\left(\frac{1}{\sqrt{\sigma}}f\left(x\sqrt{\frac{H\sigma}{\phi_0}}, \frac{\varepsilon}{\pi\xi\Delta_0}\sqrt{\frac{\sigma\phi_0}{H}}\right) + \sqrt{\sigma}f\left(y\sqrt{\frac{H}{\sigma\phi_0}}, \frac{\varepsilon}{\pi\xi\Delta_0}\sqrt{\frac{\phi_0}{\sigma H}}\right)\right) \quad (18)$$

$$N_{II}(\varepsilon, x, y) = \frac{N_F}{4}\sqrt{\frac{\pi H}{2H_{c2}}}\left(\frac{1}{2\sqrt{\sigma}}f\left(2x\sqrt{\frac{H\sigma}{\phi_0}}, \frac{2\varepsilon}{\pi\xi\Delta_0}\sqrt{\frac{\sigma\phi_0}{H}}\right) + \sqrt{\sigma}f\left(y\sqrt{\frac{H}{\sigma\phi_0}}, \frac{\varepsilon}{\pi\xi\Delta_0}\sqrt{\frac{\phi_0}{\sigma H}}\right)\right) \quad (19)$$

$$f(z, \tilde{\varepsilon}) = |\Phi(z)/2 - \tilde{\varepsilon}| + |\Phi(z)/2 + \tilde{\varepsilon}|$$

where  $N_I$  ( $N_{II}$ ) is the local DOS for the lattices of the type I (II),  $N_F$  is the density of states at the Fermi level in a normal state. Contour plots of the functions  $4N_I(x, y)N_F^{-1}\sqrt{2H_{c2}/(\pi H)}$  (for the rectangular lattice with  $\sigma = 2$ ) and  $4N_{II}(x, y)N_F^{-1}\sqrt{2H_{c2}/(\pi H)}$  (for the hexagonal lattice with  $\sigma = \sqrt{3}/2$ ) are shown in Fig. 2 and Fig. 3, respectively. The peaks in the local DOS occur at the points of intersection of lines parallel to the  $x$  and  $y$  axes and passing through the vortex centers. The interesting fact is that for type II lattices the peaks in the local DOS appear not only at the vortex centers while for rectangular lattices (type I) the coordinates of all peaks coincide with the vortex positions. In the vicinity of each vortex center ( $\bar{x}, \bar{y}$ ) the local DOS exhibits a fourfold

symmetry which is in a good agreement with the previous studies of a single isolated vortex in a d-wave superconductor [15,16] and decreases linearly with an increase of distances  $|x - \bar{x}|$  and  $|y - \bar{y}|$  from the vortex center (contrary to the  $((x - \bar{x})^2 + (y - \bar{y})^2)^{-1/2}$  divergence resulting from the semiclassical approach based on Eq. (1)). The spatially averaged DOS has the form

$$\langle N_I \rangle = \frac{N_F}{8} \sqrt{\frac{\pi H}{2H_{c2}}} \left( \sqrt{\sigma} + \frac{1}{\sqrt{\sigma}} \right) \left( 1 + \frac{4\varepsilon^2\phi_0}{\pi^2\xi^2\Delta_0^2 H} \right) \quad (20)$$

$$\langle N_{II} \rangle = \frac{N_F}{8} \sqrt{\frac{\pi H}{2H_{c2}}} \left( \sqrt{\sigma} + \frac{1}{2\sqrt{\sigma}} \right) \left( 1 + \frac{8\varepsilon^2\phi_0}{\pi^2\xi^2\Delta_0^2 H} \right) \quad (21)$$

If we assume the parameter  $\sigma$  to be field independent then the magnetic field dependence of the residual DOS (at  $\varepsilon = 0$ ) follows the square root behavior which was first predicted in Ref. [3]. The deviations from this behaviour may appear if the vortex lattice structure and, hence, the  $\sigma$  value change with a magnetic field increasing (see Refs. [23,24,26–28]) and, probably, they are the cause for the discrepancies in specific heat measurements [6,7]. Note that these discrepancies may be also explained taking account of the disorder effects [4,5].

In summary, we have described the distinctive features of the low lying quasiparticle states in the vortex lattices in d-wave systems. The periodic superfluid velocity field is shown to induce the band structure in the low energy part of the spectrum. This band structure and corresponding eigenfunctions have been analysed for both rectangular and centered rectangular flux lattices tilted by  $\pi/4$  from the  $a$  axis. The resulting peculiarities of the local DOS are shown to be strongly influenced by the vortex lattice geometry. The unusual behaviour of DOS discussed above can be probed by specific heat and STM measurements and could provide additional arguments in favour of the d-wave pairing in high- $T_c$  superconductors. The present study provides a starting point for the analysis of static and dynamic properties of the mixed state in various d-wave systems, including, probably, high- $T_c$  copper oxides. For this purpose the approach developed above should be generalized to describe the quasiparticle states for arbitrary lattice orientations with respect to crystal axes.

It is my pleasure to thank Dr.I.D.Tokman for very useful discussions of the issues considered in this article. This work was supported, in part, by the Russian Foundation for Fundamental Research (Grant No. 97-02-17437).

## REFERENCES

- [1] J.Annett, N.Goldenfeld, and A.J.Leggett, Report No. cond-mat/9601060 [”Physical Properties of High Temperature Superconductors”, edited by D.M.Ginsberg (World Scientific, Singapore, 1996), Vol.5].
- [2] C.Caroli, P.G. de Gennes, J.Matrimon, Phys.Lett. **9**, 307 (1964).
- [3] G.E.Volovik, Pis’ma Zh.Eksp.Teor.Fiz. **58**, 457 (1993) [JETP Lett.**58**, 469 (1993)].
- [4] Yu.S.Barash, V.P.Mineev, and A.A.Svidzinskii, Pis’ma Zh.Eksp.Teor.Fiz. **65**, 606 (1997) [JETP Lett.**65**, 638 (1997)].
- [5] C.Kübert and P.J.Hirschfeld, Report No. cond-mat/9708221.
- [6] K.A.Moler *et al.*, Phys.Rev.Lett.**73**, 2744 (1994); Phys.Rev.B **55**, 3954 (1997).
- [7] R.A.Revaz *et al.*, Czech J. Phys. **46**, Suppl., part S3, 1205 (1996).
- [8] I.Maggio-Aprile *et al.*, Phys.Rev.Lett.**75**, 2754 (1995).
- [9] N.B.Kopnin and G.E.Volovik, Pis’ma Zh.Eksp.Teor.Fiz. **64**, 641 (1996) [JETP Lett.**64**, 690 (1996)].
- [10] N.B.Kopnin and G.E.Volovik, Phys.Rev.Lett. **79**, 1377 (1997).
- [11] N.B.Kopnin, Report No. cond-mat/9711246.
- [12] Yu.G.Makhlin, Phys.Rev.B **56**, 11872 (1997).
- [13] S.H.Simon and P.A.Lee, Phys.Rev.Lett. **78**, 1548 (1997).
- [14] M.Franz and Z.Tešanović, Report No. cond-mat/9710258.
- [15] N.Schopohl and K.Maki, Phys.Rev.B **52**, 490 (1995).
- [16] M.Ichioka *et al.*, Phys.Rev.B **53**, 15316 (1996).
- [17] P.I.Soininen, C.Kallin, and A.J.Berlinsky, Phys.Rev.B **50**, 13883 (1994).
- [18] Y.Wang and A.H.MacDonald, Phys.Rev.B **52**, R3876 (1995).
- [19] Y.Morita, M.Kohmoto, and K.Maki, Report No. cond-mat/9706118.
- [20] C.Bruder, Phys.Rev.B **41**, 4017 (1990).
- [21] G.E.Volovik and N.B.Kopnin, Phys.Rev.Lett. **78**, 5028 (1997).
- [22] S.H.Simon and P.A.Lee, Phys.Rev.Lett. **78**, 5029 (1997).
- [23] M.Franz, I.Affleck, and M.H.S.Amin, Phys.Rev.Lett. **79**, 1555 (1997).
- [24] M.H.S.Amin, I.Affleck, and M.Franz, Report No. cond-mat/9712218.
- [25] H.Won and K.Maki, Phys.Rev.B **53**, 5927 (1996).
- [26] J.Shiraishi, M.Kohmoto, and K.Maki, Report No. cond-mat/9709310.
- [27] J.Shiraishi, M.Kohmoto, and K.Maki, Report No. cond-mat/9802067.
- [28] M.Ichioka, N.Enomoto, and K.Machida, Report No. cond-mat/9704147.
- [29] B.Keimer *et al.*, Phys.Rev.Lett.**73**, 3459 (1994).
- [30] E.O.Kane and E.I.Blount, in ”Tunneling Phenomena in Solids”, edited by E.Burstein and S.Lundqvist (Plenum Press, New York, 1969).
- [31] E.T.Whittaker and G.N.Watson, Modern Analysis (Cambridge University Press, 1947).

## FIGURES

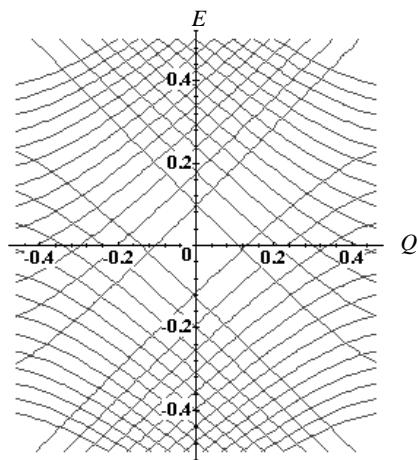


FIG. 1. The energy branches in the first Brillouin zone for  $\lambda = 0.01$ .

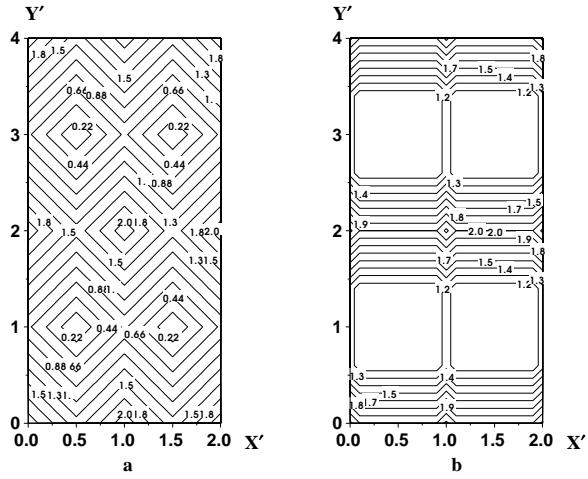


FIG. 2. Contour plot of the local DOS for the rectangular lattice with  $\sigma = 2$  ( $x' = x(2H/\phi_0)^{0.5}$ ,  $y' = y(2H/\phi_0)^{0.5}$ ): (a) $\varepsilon = 0$ ; (b) $\varepsilon = 0.2\pi\xi\Delta_0(2H/\phi_0)^{0.5}$ .

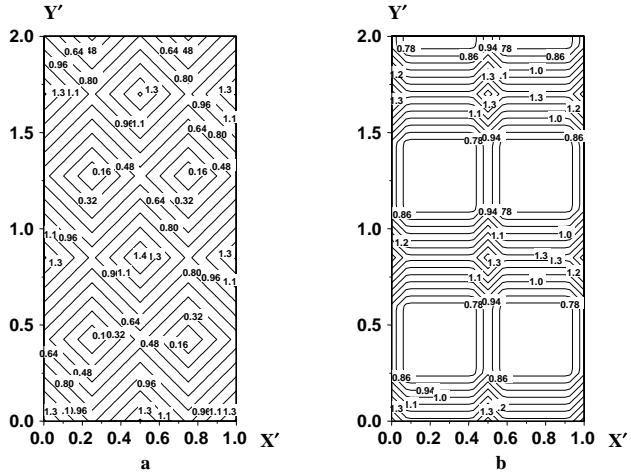


FIG. 3. Contour plot of the local DOS for the hexagonal lattice: ( $x' = x(\sigma H/\phi_0)^{0.5}$ ,  $y' = y(\sigma H/\phi_0)^{0.5}$ ): (a) $\varepsilon = 0$ ; (b) $\varepsilon = 0.2\pi\xi\Delta_0(\sigma H/\phi_0)^{0.5}$ .